Probabilistic functions and cryptographic oracles in higher order logic

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Abstract. This paper presents a shallow embedding of a probabilistic functional programming language in higher order logic. The language features monadic sequencing, recursion, random sampling, failures and failure handling, and black-box access to oracles. Oracles are probabilistic functions which maintain hidden state between different invocations. To that end, we propose generative probabilistic systems as the semantic domain in which the operators of the language are defined. We prove that these operators are parametric and derive a relational program logic for reasoning about programs from parametricity. Several examples demonstrate that our language is suitable for conducting cryptographic proofs.

1 Introduction

As cryptographic algorithms and protocols are becoming increasingly complicated, flaws in their security proofs are more likely to be overlooked. So, security vulnerabilities can remain undiscovered. The game playing technique [16,29,48] helps in structuring such proofs and can thus increase the level of confidence in their correctness. In this view, security notions are expressed as programs (called games) in the same language as the cryptographic algorithms and protocols. The proof then consists of transforming the program in several small steps until the desired properties obviously hold. To achieve high assurance, several frameworks [8,14,42] and a proof assistant [13] offer machine support in formalising algorithms and security notions and in checking such sequences of game transformations. This way, many cryptographic constructions have been mechanically proven secure, e.g., [6,7,43].

For security protocols such as TLS, Kerberos, and IPSec, only few mechanised security proofs in the computational model are available, e.g., [18,21]. Instead, symbolic analysis tools [4,19,39,47] dominate. They model protocol messages as terms in an algebra rather than bitstrings and assume that cryptography is perfect. Computational soundness (CS) results [1,9,25] bridge the gap between symbolic and computational models, but to our knowledge, they have never been mechanically checked. Yet, mechanising them is desirable for three reasons. First, the proofs are extremely technical with many case distinctions. A proof assistant can check that all cases are covered and no assumption is forgotten (as has happened in early CS works), and it provides automation for dealing with the easy cases. Second, computational soundness results need to be formalised only once per set of cryptographic primitives, not for every protocol to which the symbolic analyser is applied. That is, one mechanised proof yields trustworthy proofs for a whole class of protocols. Third, mechanisation supports the evolution of models and proofs. If a primitive is added or an assumption weakened as in [10], the proof assistant checks the proofs again and pinpoints where adaptations are needed.

Unfortunately, the existing frameworks are not suitable for formalising CS proofs. CertiCrypt [14] and Verypto [8] formalise stateful languages in the proof assistants Coq and Isabelle, respectively. Due to the deep embedding, program transformations always require a proof about the semantics and programs cannot easily reuse the existing libraries of the proof assistants. Therefore, formalising cryptographic arguments requires a very substantial effort. CertiCrypt's successor EasyCrypt [13] provides much better automation using SMT solvers. Its logical foundation is higher order logic (HOL), but the reasoning infrastructure focuses on proofs in relational Hoare logic. However, substantial parts of CS proofs reason about the term algebra of the symbolic model for which EasyCrypt provides no support at all. The foundational cryptography framework (FCF) [42] alleviates CertiCrypt's formalisation burden by taking a semi-shallow approach in Coq. For pure deterministic functions, the framework reuses the language of the logic; only probabilistic effects and interaction with oracles are modelled syntactically as monads. This saves considerable effort in comparison to CertiCrypt. Yet, the monadic language lacks features such as recursion and exceptions, which are desirable for CS, e.g., to implement probabilistic serialisers and parsers.

Contributions. We present a framework for cryptographic proofs formalised in higher order logic. It consists of a probabilistic functional programming language and a logic for reasoning about programs and probabilities. The language features monadic sequencing, recursion, random sampling, failures and their handling, and black-box access to oracles. Oracles are probabilistic functions that maintain hidden state between different invocations. We model the language shallowly, i.e., we define the operators directly in the semantic domain. Programs which need no access to oracles are interpreted in the monad of discrete subprobabilities (§2.2); recursive functions are defined in terms of a least fixpoint operator (§2.3). For programs with oracle access, we propose *generative probabilistic systems* as semantic domain, which supports corecursive definitions (§3.2). In particular, we define operators for composing programs with oracles and other programs. We have implemented the framework in the proof assistant Isabelle/HOL [41], but it could be formalised in any other HOL-based prover, too.

The shallow embedding offers three benefits. First, we can reuse all the existing infrastructure of the proof assistant, such as name binding, higher-order unification and the module system, but also definitional packages and existing libraries. Second, we obtain a rich equational theory directly on programs, i.e., without any intervening interpretation function. Equality is important as we can replace terms by equal terms in any context (by HOL's substitution rule), i.e., there is no need for congruences. Consequently, proof automation can use the (conditional) equations directly for rewriting. Although no syntax is formalised in the logic, programs are still written as HOL terms. This suffices to guide syntax-directed proof tactics. Third, the language is not restricted to a fixed set of primitives. New operators can be defined in the semantic domain at any time.

Beyond equality, we provide a relational logic for reasoning about programs (§2.4 and §3.3). Relational parametricity [44,50] has been our guide in that most rules follow from the fact that the operators are parametric. In particular, we demonstrate that several common reasoning principles in cryptographic proofs follow from parametricity. This approach ensures soundness of the logic by construction and can also guide the discovery of proof rules for new operators. Our logic is similar to those of the other tools. The novelty is that we follow a principled way in finding the rules and establishing soundness.

Three examples demonstrate that our framework is suitable for cryptographic proofs. We proved indistinguishability under chosen plaintext attacks (IND-CPA) for (i) Elgamal public-key encryption [27] in the standard model (§2.1 and §2.5), (ii) Hashed Elgamal encryption in the random oracle model (§3.1 and §3.4), and (iii) an encryption scheme based on pseudo-random functions [42]. The examples have been chosen to enable comparisons with the existing frameworks (see §4). They show that our framework leads to concise proofs with the level of proof automation being comparable to EasyCrypt's, the current state of the art. This indicates that the framework scales to computational soundness results, although our examples are much simpler. Indeed, we have just started formalising a CS result, so this is only a first step. The framework and all examples and proofs have been formalised in Isabelle/HOL and are available online [37].

Preliminaries: HOL notation The meta-language HOL mostly uses everyday mathematical notation. Here, we present basic non-standard notation and a few types with their operations; further concepts will be introduced when needed.

HOL terms are simply typed lambda terms with let-polymorphism (we use Greek letters $\alpha, \beta, \ldots$ for type variables). Types include in particular the type of truth values bool and the singleton type unit with its only element $()$ and the space of total functions $\alpha \Rightarrow \beta$. Type constructors are normally written postfix, e.g., $\mathsf{bool}\ \mathsf{list}$ denotes the type of finite lists of booleans, i.e., bitstrings. The notation $t :: \tau$ means that the HOL term t has type τ.

Pairs (type $\alpha \times \beta$) come with two projection functions π_1 and π_2, and the map function $\mathsf{map}_\times\ f\ g\ (a, b) = (f\ a, g\ b)$. Tuples are identified with pairs nested to the right, i.e., (a, b, c) is identical to $(a, (b, c))$ and $\alpha \times \beta \times \gamma$ to $\alpha \times (\beta \times \gamma)$. Dually, $\alpha + \beta$ denotes the disjoint sum of α and β; the injections are $\mathsf{Inl} :: \alpha \Rightarrow \alpha + \beta$ and $\mathsf{Inr} :: \beta \Rightarrow \alpha + \beta$. Case distinctions on freely generated types use guard-like syntax. The map function map_+ for disjoint sums, e.g., pattern matches on x to apply the appropriate function: $\mathsf{map}_+\ f\ g\ x = \mathsf{case}\ x\ \mathsf{of}\ \mathsf{Inl}\ y \Rightarrow \mathsf{Inl}\ (f\ y) \mid \mathsf{Inr}\ z \Rightarrow \mathsf{Inr}\ (g\ z)$.

Sets (type $\alpha\ \mathsf{set}$) are isomorphic to predicates (type $\alpha \Rightarrow \mathsf{bool}$) via the bijections membership $\in$ and set comprehension $\{\, x.\ \_\}$; the empty set is $\{\,\}$. Binary relations are sets of pairs and written infix, i.e., $x\ R\ y$ denotes $(x, y) \in R$. The relators $\mathsf{rel}_\times$ and rel_+ lift relations component-wise to pairs and sums.

The datatype $\alpha\ \mathsf{option} = \mathsf{None} \mid \mathsf{Some}\ \alpha$ corresponds to the Haskell type `Maybe`. It adjoins a new element None to α, all existing values in α are prefixed by Some. Maps (partial functions) are modelled as functions of type $\alpha \Rightarrow \beta$ option, where None represents undefinedness and $f\ x = \mathsf{Some}\ y$ means that f maps x to y. The empty map $\varnothing = (\lambda\_.\ \mathsf{None})$ is undefined everywhere. Map update is defined as follows: $f(a \mapsto b) = (\lambda x.\ \mathsf{if}\ x = a\ \mathsf{then}\ \mathsf{Some}\ b\ \mathsf{else}\ f\ x)$.

2 A Shallow Probabilistic Functional Language

Security notions in the computation model are expressed as games parametrised by an adversary. In formalising such games, we want to leverage as much of the prover's infrastructure as possible. Therefore, we only model explicitly what cannot be expressed in the prover's term language, namely probabilities and access to oracles. In this section, we focus on probabilities and show the framework in action on the example of Elgamal encryption (§2.1 and §2.5).

We model games as functions which return a discrete (sub)probability distribution over outcomes. Discrete subprobabilities strike a balance between expressiveness and ease of use. They provide a monadic structure for sequencing and failure (§2.2). Thus, games can be formulated naturally and control may be transferred non-locally in error cases such as invalid data produced by the adversary. They also host a fixpoint operator for defining recursive functions (§2.3). In contrast, measure-theoretic (sub)probability distributions clutter proofs with measurability requirements. For computational soundness, discrete subprobability distributions suffice. In §2.4, we prove that the operators are relationally parametric and derive a programming logic and common cryptographic reasoning principles from parametricity.

2.1 Example: Elgamal Encryption

In this section, we formalise Elgamal's encryption scheme [27] to motivate the features of our language. In §2.5, we prove the scheme IND-CPA secure under the decisional Diffie-Hellman (DDH) assumption. We formally introduce the language only in §2.2. For now, an intuitive understanding suffices: monadic sequencing is written in Haskell-style do notation and $x \leftarrow \mathsf{uniform}\ A$ samples x as a random element from the finite set A.

For the following, consider a fixed finite cyclic group $\mathcal{G}$ over the type α with generator g. We write $\otimes$ for group multiplication and $\hat{}$ for exponentiation with natural numbers; $|\mathcal{G}|$ denotes the order of $\mathcal{G}$. In Elgamal, the public key is an arbitrary group element $\mathsf{g}\,\hat{}\,x$ and the private key is the exponent x. The security of this scheme relies on the hardness of computing the discrete logarithm. The key generation algorithm $\mathsf{key\text{-}gen}$ generates a new key pair by randomly sampling the exponent (Fig. 1b). Messages are group elements, too. To encrypt a message m under the public key α, the algorithm multiplies m with α raised to a random power between 0 and the order of the group (Fig. 1a).

Elgamal's encryption scheme produces ciphertexts that are indistinguishable under chosen plaintext attacks. Chosen plaintext attacks are formalised as the

```
aenc $\alpha$ $m$ = do {
    $y \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    return$_{\mathsf{spmf}}$ $(\mathsf{g} \,\hat{}\, y, (\alpha \,\hat{}\, y) \otimes m)$ }
```

(a) Elgamal encryption algorithm

```
key-gen = do {
    $x \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    return$_{\mathsf{spmf}}$ $(\mathsf{g} \,\hat{}\, x, x)$ }
```

(b) Elgamal key generation

```
ind-cpa $(\mathcal{A}_1, \mathcal{A}_2)$ = try do {
    $(pk, sk) \leftarrow$ key-gen;
    $b \leftarrow$ coin;
    $((m_0, m_1), \sigma) \leftarrow \mathcal{A}_1\ pk$;
    assert (valid-plain $m_0 \wedge$ valid-plain $m_1$);
    $c^* \leftarrow$ aenc $pk$ (if $b$ then $m_0$ else $m_1$);
    $b' \leftarrow \mathcal{A}_2\ c^*\ \sigma$;
    return$_{\mathsf{spmf}}$ $(b = b')$
} else coin
```

(c) IND-CPA security game without oracles

Fig. 1: Examples of cryptographic algorithms and games without oracle access

game ind-cpa shown in Fig. 1c. An IND-CPA adversary $\mathcal{A}$ consists of two probabilistic functions $\mathcal{A}_1$ and $\mathcal{A}_2$. Given a public key pk, $\mathcal{A}_1$ chooses two plaintexts m_0 and m_1. The game then encrypts one of them as determined by the random bit b (a coin flip) and gives the challenge ciphertext c^* to $\mathcal{A}_2$ and any arbitrary state information σ produced by $\mathcal{A}_1$. Then, $\mathcal{A}_2$ produces a guess which of the two messages c^* decrypts to. Indistinguishability requires that the adversary cannot do significantly better than flipping a coin. This is measured by the IND-CPA advantage given by adv-ind-cpa $\mathcal{A} = |$ind-cpa $\mathcal{A}$! True $- 1/2|$. A concrete security theorem bounds the advantage by a quantity which is known or assumed to be small.

If any step in the game fails, ind-cpa behaves like a fair coin flip, i.e., the advantage is 0 in that case. This happens, e.g., if the plaintexts are invalid, i.e., not elements of the group, or the adversary does not produce plaintexts or a guess at all. (In an implementation, the latter could be detected using timeouts.)

The DDH assumption states that given two random group elements $\mathsf{g} \,\hat{}\, x$ and $\mathsf{g} \,\hat{}\, y$, it is hard to distinguish $\mathsf{g} \,\hat{}\, (x \cdot y)$ from another random group element $\mathsf{g} \,\hat{}\, z$. Formally, a DDH adversary $\mathcal{A}$ is a probabilistic function that takes three group elements and outputs a Boolean. We model the two settings as two games ddh_0 and ddh_1 parametrised by the adversary.

```
ddh$_0$ $\mathcal{A}$ = do {
    $x \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    $y \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;

    $\mathcal{A}\ (\mathsf{g} \,\hat{}\, x)\ (\mathsf{g} \,\hat{}\, y)\ (\mathsf{g} \,\hat{}\, (x \cdot y))$ }
```

```
ddh$_1$ $\mathcal{A}$ = do {
    $x \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    $y \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    $z \leftarrow$ uniform $\{\, 0 ..< |\mathcal{G}| \,\}$;
    $\mathcal{A}\ (\mathsf{g} \,\hat{}\, x)\ (\mathsf{g} \,\hat{}\, y)\ (\mathsf{g} \,\hat{}\, z)$ }
```

The DDH advantage captures the difficulty of $\mathcal{A}$ distinguishing the two settings. It is defined as adv-ddh $\mathcal{A} = |\mathsf{ddh}_0\ \mathcal{A}$! True $- \mathsf{ddh}_1\ \mathcal{A}$! True$|$. The DDH assumption states that the advantage is small, and in §2.5, we show that the IND-CPA advantage for Elgamal is bounded by the DDH advantage.

2.2 The Monad of Discrete Subprobability Distributions

A discrete subprobability distribution is given by its subprobability mass function (spmf), i.e., a non-negative real-valued function which sums up to *at most* 1.

We define the type $\alpha\ \mathsf{spmf}$ of all spmfs[1] over elementary events of type α and use variables p, q for spmfs. We make applications of spmfs explicit using the operator !. So, $p \,!\, x$ denotes the subprobability mass that the spmf p assigns to the elementary event x. An event A is a set of elementary events; its subprobability $\mathsf{measure}\ p\ A$ is given by $\sum_{y \in A} p \,!\, y$. Moreover, the weight $\|p\|$ of p is the total probability mass assigned by p, i.e., $\|p\| = \sum_y p \,!\, y$. If p is a probability distribution, i.e., $\|p\| = 1$, then we call p lossless following [14] (notation $\mathsf{lossless}\ p$). The support $\mathsf{set}_{\mathsf{spmf}}\ p = \{\, x.\ p \,!\, x > 0 \,\}$ is countable by construction.

The type $\alpha\ \mathsf{spmf}$ hosts the polymorphic monad operations $\mathsf{return}_{\mathsf{spmf}} :: \alpha \Rightarrow \alpha\ \mathsf{spmf}$ and $\mathsf{bind}_{\mathsf{spmf}} :: \alpha\ \mathsf{spmf} \Rightarrow (\alpha \Rightarrow \beta\ \mathsf{spmf}) \Rightarrow \beta\ \mathsf{spmf}$ given by

$$\mathsf{return}_{\mathsf{spmf}}\ y \,!\, x = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \qquad \mathsf{bind}_{\mathsf{spmf}}\ p\ f \,!\, x = \sum_{y \in \mathsf{set}_{\mathsf{spmf}}\ p} (p \,!\, y) \cdot (f\ y \,!\, x)$$

In this paper and in our formalisation, we use Haskell-style do notation where $\mathbf{do}\ \{\ x \leftarrow p;\ f\ \}$ desugars to $\mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda x.\ \mathbf{do}\ f)$. The monad operations satisfy the usual monad laws: (i) $\mathsf{bind}_{\mathsf{spmf}}$ is associative and (ii) $\mathsf{return}_{\mathsf{spmf}}$ is neutral for $\mathsf{bind}_{\mathsf{spmf}}$. In addition, $\mathsf{bind}_{\mathsf{spmf}}$ is commutative and constant elements cancel.

$$\mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda x.\ \mathsf{bind}_{\mathsf{spmf}}\ q\ (f\ x)) = \mathsf{bind}_{\mathsf{spmf}}\ q\ (\lambda y.\ \mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda x.\ f\ x\ y)) \tag{1}$$

$$\mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda\_.\ q) = \mathsf{scale}\ \|p\|\ q \tag{2}$$

Here, $\mathsf{scale}\ r\ p$ scales the subprobability masses of p by r, i.e., $\mathsf{scale}\ r\ p!x = r \cdot (p!x)$ for $0 \leq r \leq 1/\|p\|$. In particular, if p is lossless, then $\mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda\_.\ q) = q$. The monad operations give rise to the functorial action $\mathsf{map}_{\mathsf{spmf}} :: (\alpha \Rightarrow \beta) \Rightarrow \alpha\ \mathsf{spmf} \Rightarrow \beta\ \mathsf{spmf}$ given by $\mathsf{map}_{\mathsf{spmf}}\ f\ p\ = \mathsf{bind}_{\mathsf{spmf}}\ p\ (\lambda x.\ \mathsf{return}_{\mathsf{spmf}}\ (f\ x))$.

For sampling, the monad provides an operation $\mathsf{uniform}$ which returns the uniform distribution over a finite set. There are three special cases worth mentioning. First, for a singleton set, we have $\mathsf{uniform}\ \{\, x \,\} = \mathsf{return}_{\mathsf{spmf}}\ x$. Second, as fair coin flips are particularly prominent in cryptographic games, we abbreviate $\mathsf{uniform}\ \{\, \mathsf{True}, \mathsf{False} \,\}$ with coin. Third, in case of an empty set, we let $\mathsf{uniform}\ \{\}$ denote the empty subprobability distribution $\bot$ which assigns no probability to any event at all, i.e., $\bot \,!\, x = 0$ for all x. In combination with sequencing and recursion, $\mathsf{uniform}$ is fairly expressive.

For example, the Bernoulli distribution $\mathsf{bernoulli}\ r$, which returns True with probability r for any $0 \leq r \leq 1$, can be sampled from fair coin flips as shown on the right [32]. It can be used to define probabilistic choice.

```
bernoulli $r = \mathbf{do}$ {
  $b \leftarrow$ coin;
  if $b$ then $\mathsf{return}_{\mathsf{spmf}}\ (r \geq 1/2)$
  else if $r < 1/2$ then bernoulli $(2 \cdot r)$
  else bernoulli $(2 \cdot r - 1)$ }
```

Fig. 2: The Bernoulli distribution

[1] In the formalisation, we construct the type $\alpha\ \mathsf{spmf}$ by combining the existing monad for probability mass functions [30] with an exception monad. This way, most of our primitive operations can be defined in terms of the primitive operations of the two monads. Hence, we can derive many of their properties, in particular parametricity (§2.4), from the latter's.

Moreover, the subprobability monad contains a failure element, namely $\bot$. Failure aborts the current part of the program, as $\bot$ propagates: $\mathsf{bind_{spmf}}\ \bot\ f = \bot$. However, we hardly use $\bot$ in programs directly. It is more natural to define an assertion statement $\mathsf{assert}\ b = \mathsf{if}\ b\ \mathsf{then}\ \mathsf{return_{spmf}}\ ()\ \mathsf{else}\ \bot$. Assertions are useful in validating the inputs received from the adversary. For example, the assertion in the IND-CPA security game in Fig. 1c checks that the adversary $\mathcal{A}_1$ produced valid plaintexts.

Failures are handled using the statement $\mathsf{try}\ p\ \mathsf{else}\ q$, which distributes the probability mass not assigned by p according to q. Formally, it satisfies $(\mathsf{try}\ p\ \mathsf{else}\ q)\,!\,x = p\,!\,x + (1 - \|p\|) \cdot q\,!\,x$ for all x. Clearly, we have the equalities $\mathsf{try}\ \bot\ \mathsf{else}\ q = q$ and $\mathsf{try}\ p\ \mathsf{else}\ q = p$ if p is lossless. Moreover, try commutes with $\mathsf{bind_{spmf}}$ for lossless spmfs, i.e., we can enlarge or shrink the scope of the handler.

$$\mathsf{lossless}\ p \longrightarrow \mathsf{try}\ \mathbf{do}\ \{\ x \leftarrow p;\ \ f\ x\ \}\ \mathsf{else}\ q = \mathbf{do}\ \{\ x \leftarrow p;\ \mathsf{try}\ f\ x\ \mathsf{else}\ q\ \}$$

$$\begin{aligned}\mathsf{lossless}\ q \longrightarrow \mathsf{try}\ \mathbf{do}\ \{\ x \leftarrow p;\ \ f\ x\ \}\ \mathsf{else}\ q =& \\ \mathsf{try}\ \mathbf{do}\ \{\ x \leftarrow p;\ \mathsf{try}\ f\ x\ \mathsf{else}\ q\ \}\ \mathsf{else}\ q&\end{aligned}$$

The IND-CPA game in Fig. 1c treats failures as fair coin flips. This is sound as the advantage is the probability of the outcome True less $1/2$.

This completes the exposition of the language primitives except for the fixpoint combinator (see §2.3). They suffice for all examples in this paper, but note that we are not restricted to this set of operations. If necessary, users can define their own discrete subprobability distribution on the spot thanks to the shallow embedding. Also remember that all the equation in this section are equalities *inside* the logic. Hence, we can use them for rewriting in any context, not just under a semantics interpretation as with a deep embedding.

2.3 Recursive Functions in the SPMF Monad

In this section, we consider the denotation of recursive functions in the spmf monad. As usual in programming languages, we interpret a recursive specification as the least fixpoint of the associated functional. In the case of spmf, the approximation order $\sqsubseteq$ is given by $p \sqsubseteq q \leftrightarrow (\forall x.\ p\,!\,x \leq q\,!\,x)$ [5]. In this order, every chain Y has a least upper bound (lub) $\bigsqcup Y$ which is taken pointwise: $\bigsqcup Y\,!\,x = \mathsf{SUP}\ \{\,p\,!\,x.\ p \in Y\,\}$ where $\mathsf{SUP}\ A$ denotes the supremum of a bounded set A of real numbers. Thus, the approximation order is a chain-complete partial order (ccpo) with least element $\bot$ (see Prop. 1 below).

By Tarski's fixpoint theorem, every monotone function f on a ccpo has a least fixpoint $\mathsf{fix}\ f$, which is obtained by the least upper bound of the transfinite iteration of f starting at the least element. Therefore, we can define recursive functions as the least fixpoint of the associated (monotone) functional.

Using Isabelle's package for recursive monadic function definitions [34], we hide the internal construction via fixpoints from the user and automate the monotonicity proof. For example, the function $\mathsf{bernoulli}$ can be specified exactly as shown in Fig. 2. The monotonicity proof succeeds as $\mathsf{bind_{spmf}}$ is monotone in both arguments. Namely, if $p \sqsubseteq q$ and $f\ x \sqsubseteq g\ x$ for all x, then $\mathsf{bind_{spmf}}\ p\ f \sqsubseteq$

$\mathsf{bind}_{\mathsf{spmf}}\ q\ g$. In contrast, $\mathsf{try}\ \_\ \mathsf{else}\ \_$ is monotone only in the second argument, but not in the first. For example, $\bot \sqsubseteq \mathsf{return}_{\mathsf{spmf}}\ 0$, but $\mathsf{try}\ \bot\ \mathsf{else}\ \mathsf{return}_{\mathsf{spmf}}\ 1 = \mathsf{return}_{\mathsf{spmf}}\ 1 \not\sqsubseteq \mathsf{return}_{\mathsf{spmf}}\ 0 = \mathsf{try}\ \mathsf{return}_{\mathsf{spmf}}\ 0\ \mathsf{else}\ \mathsf{return}_{\mathsf{spmf}}\ 1$. Therefore, recursion is always possible through $\mathsf{bind}_{\mathsf{spmf}}$ and else, but not in general through try.

Proposition 1. *The approximation order $\sqsubseteq$ is a chain-complete partial order.*

Proof. We have to show that $\sqsubseteq$ is a partial order and that $\bigsqcup Y$ is well-defined and the least upper bound for every chain Y, i.e., every set of spmfs all of whose elements are comparable in $\sqsubseteq$. The difficult part is to show that $\bigsqcup Y$ is well-defined. In particular, we must show that the support of $\bigsqcup Y$ is countable even if Y is uncountable. Then, it is not hard to see that $\bigsqcup Y$ sums up to at most 1.

Clearly, we have $\mathsf{set}_{\mathsf{spmf}}\ (\bigsqcup Y) = \bigcup_{p \in Y} \mathsf{set}_{\mathsf{spmf}}\ p$. Yet, the union of an uncountable sequence of increasing countable sets need not be countable in general. In the following, we show that even for uncountable chains Y of spmfs, the union of the supports remains countable. To that end, we identify a countable sub-sequence of Y whose lub has the same support. The key idea is that for $\sqsubseteq$-comparable spmfs p and q, the order can be decided by looking only at the assigned probability masses, namely, $p \sqsubseteq q$ iff $\|p\| \leq \|q\|$. So suppose without loss of generality that Y does not contain a maximal element (otherwise, the lub is the maximal element and we are done). The set of assigned probability masses $A = \{\, \|p\|.\ p \in Y \,\}$ has a supemum $r \leq 1$, as 1 bounds the set from above. The closure of A contains the supremum r, so A must contain a countable increasing sequence which converges to r. This sequence gives rise to a countable sub-sequence Z of Y, for which we show $(\bigcup_{p \in Y} \mathsf{set}_{\mathsf{spmf}}\ p) \subseteq (\bigcup_{q \in Z} \mathsf{set}_{\mathsf{spmf}}\ q)$. For any $p \in Y$, there is a $q \in Z$ such that $\|p\| \leq \|q\|$, as the assigned probability masses in Z converge to r from below and p is not maximal. Hence, $p \sqsubseteq q$ as p and q are related in $\sqsubseteq$, and therefore $\mathsf{set}_{\mathsf{spmf}}\ p \subseteq \mathsf{set}_{\mathsf{spmf}}\ q$ as $\mathsf{set}_{\mathsf{spmf}}$ is monotone.

The attentive reader might wonder why we need transfinite iteration for the fixpoint despite having shown that uncountable chains can be reduced to countable ones for the purpose of lubs. Countable fixpoint iteration, which defines the least fixpoint as $\bigsqcup\{\, f^i\ \bot.\ i \in \mathbb{N} \,\}$, does not suffice. (Here, function iteration is defined by $f^0 = \mathsf{id}$ and $f^{n+1} = f \circ f^n$.) The reason is that the chain $\{\, f^i\ \bot.\ i \in \mathbb{N} \,\}$ might stop before the least fixpoint is reached. Consider, e.g., the monotone spmf transformer $f :: \mathsf{unit\ spmf} \Rightarrow \mathsf{unit\ spmf}$ given below.

$$f\ p\ !\ x = \mathsf{if}\ p\ !\ x < \tfrac{1}{2}\ \mathsf{then}\ \tfrac{2 \cdot p!x+1}{4}\ \mathsf{else}\ 1$$

The countable iteration of f starting at $\bot$ yields a sequence of spmfs which assign to () the masses 0, $1/4$, $3/8$, $7/16$, $15/32$, ... The least upper bound of this sequence assigns $1/2$ to (). That is, the iteration has not yet reached f's fixed point, which assigns the mass 1 to (). This is because f is not (chain) continuous, i.e., it does not preserve lubs.

Overall, arbitrary chains and transfinite iteration are superior to ordinary fixpoint iteration in two ways. First, our fixpoint combinator can handle more functions, i.e., we can accept more recursive specifications. Second, the proof obligations that recursive specifications incur are simpler: monotonicity is usually easier to show than continuity.

2.4 Lifting and Parametricity

Game playing proofs transform games step by step. In each step, we have to bound the probability that the adversary can distinguish the original game from the transformed one. For some transformations, the equational theory suffices to prove the games equal—then, the probability is 0. Other transformations are justified by cryptographic assumptions. To bound the probability in such cases, our framework provides a relational logic for programs.

To that end, we first define an operation to lift relations over elementary events to relations over spmfs. With this lifting operator, our primitive operations are parametric. Then, we can derive the logic from parametricity.

Lifting The lifting operation $\mathsf{rel}_{\mathsf{spmf}}$ transforms a binary relation R over elementary events into a relation over spmfs over these events. For lossless distributions, a number of definitions have appeared in the literature. We generalise the one from [30] as the relator associated with the natural transformation $\mathsf{set}_{\mathsf{spmf}}$ [45]. Formally, $\mathsf{rel}_{\mathsf{spmf}}\ R$ relates the spmfs $p :: \alpha\ \mathsf{spmf}$ and $q :: \beta\ \mathsf{spmf}$ iff there is an spmf $w :: (\alpha \times \beta)\ \mathsf{spmf}$ such that (i) $\mathsf{set}_{\mathsf{spmf}}\ w \subseteq R$, (ii) $\mathsf{map}_{\mathsf{spmf}}\ \pi_1\ w = p$, and (iii) $\mathsf{map}_{\mathsf{spmf}}\ \pi_2\ w = q$, We call w a R-joint spmf of p and q.

This definition reformulates the one by Larsen and Skou [36] for lossless spmfs. They consider w as a non-negative weight function on the relation such that the marginals are the original distribution., i.e., w must satisfy (i) $x\ R\ y$ whenever $w!(x, y) > 0$, (ii) $\sum_y w!(x, y) = p!x$ for all x, and (iii) $\sum_x w!(x, y) = q!y$ for all y, Using the functorial structure of spmfs, our definition expresses the same conditions more abstractly without summations. In previous work [30], this led to considerably shorter proofs.

Recently, Sack and Zhang [46] showed that if p and q are lossless, then

$$p\ (\mathsf{rel}_{\mathsf{spmf}}\ R)\ q \quad \text{iff} \quad \forall A.\ \mathsf{measure}\ p\ A \le \mathsf{measure}\ q\ \{\, y.\ \exists x \in A.\ x\ R\ y, \}$$

generalising Desharnais' proof for the finite case [26]. In our formalisation, we assume the "if" direction of the equivalence. We have not yet proved it formally, as it relies on the max-flow min-cut theorem for countable networks [3], which itself requires a substantial formalisation effort. Still, we formally derive the following characterisation of $\mathsf{rel}_{\mathsf{spmf}}\ R$ for arbitrary spmfs from this assumption.

Lemma 1 (Characterisation of $\mathsf{rel}_{\mathsf{spmf}}$).
The following are equivalent for all R, p, and q.

(a) $p\ (\mathsf{rel}_{\mathsf{spmf}}\ R)\ q$
(b) $\mathsf{set}_{\mathsf{spmf}}\ w \subseteq R$ *and* $\mathsf{map}_{\mathsf{spmf}}\ \pi_1\ w = p$ *and* $\mathsf{map}_{\mathsf{spmf}}\ \pi_2\ w = q$ *for some* w
(c) $\mathsf{measure}\ p\ A \le \mathsf{measure}\ q\ \{\, y.\ \exists x \in A.\ x\ R\ y \}$ *for all* A *and* $\|p\| \ge \|q\|$

The relator enjoys a number of useful properties. For example, (i) it generalises equality, namely $\mathsf{rel}_{\mathsf{spmf}}\ (=) = (=)$, (ii) it distributes over relation composition, and (iii) it is monotone: $p\ (\mathsf{rel}_{\mathsf{spmf}}\ R)\ q$ implies $p\ (\mathsf{rel}_{\mathsf{spmf}}\ R')\ q$ if $x\ R'\ y$ whenever $x\ R\ y$ and $x \in \mathsf{set}_{\mathsf{spmf}}\ p$ and $y \in \mathsf{set}_{\mathsf{spmf}}\ q$.

Parametricity The program logic describes the interaction between the spmf operations and the relator. As it turns out, relational parametricity [44,50] helps us to find the rules and prove them sound.[2]

Parametricity expresses that a polymorphic function does not inspect the values of type variables, but works uniformly for all instances. Relational parametricity formalises this as follows. Types are interpreted as relations and type constructors as relation transformers. A polymorphic function is (relationally) parametric iff it is related to itself in its type's relation where type variables are interpreted as arbitrary relations. For spmfs, the relator $\mathsf{rel}_{\mathsf{spmf}}$ is the appropriate relation transformers. For example, parametricity for $\mathsf{return}_{\mathsf{spmf}} :: \alpha \Rightarrow \alpha\ \mathsf{spmf}$ is expressed as $\mathsf{return}_{\mathsf{spmf}}\ (R \Mapsto \mathsf{rel}_{\mathsf{spmf}}\ R)\ \mathsf{return}_{\mathsf{spmf}}$ for all $R :: (\alpha \times \beta)\ \mathsf{set}$. Here, the relator $R \Mapsto S$ for the function space relates two functions f and g iff they transform relatedness in R into relatedness in S, i.e., $x\ R\ y$ implies $(f\ x)\ S\ (g\ y)$ for all x and y.

Wadler [50] proved that all functions of the polymorphic lambda calculus are parametric. He also demonstrated that adding polymorphic equality destroys this property. Higher order logic has polymorphic equality $(=)$ and description operators, so not all HOL functions are parametric. Thus, parametricity is not a free theorem in our setting; we have to prove it. For $\mathsf{return}_{\mathsf{spmf}}$, parametricity follows directly from unfolding the definitions and taking $\mathsf{return}_{\mathsf{spmf}}\ (x, y)$ as the joint spmf for $\mathsf{return}_{\mathsf{spmf}}\ x$ and $\mathsf{return}_{\mathsf{spmf}}\ y$ for all $x\ A\ y$. For $\mathsf{bind}_{\mathsf{spmf}}$, the parametricity statement is

$$\forall R\ R'.\ \mathsf{bind}_{\mathsf{spmf}}\ (\mathsf{rel}_{\mathsf{spmf}}\ R \Mapsto (R \Mapsto \mathsf{rel}_{\mathsf{spmf}}\ R') \Mapsto \mathsf{rel}_{\mathsf{spmf}}\ R')\ \mathsf{bind}_{\mathsf{spmf}}$$

The proof is similar to the one for $\mathsf{return}_{\mathsf{spmf}}$: after having unfolded the definitions, we take $\mathsf{bind}_{\mathsf{spmf}}\ w\ h$ as the joint spmf for $\mathsf{bind}_{\mathsf{spmf}}\ p\ f$ and $\mathsf{bind}_{\mathsf{spmf}}\ q\ g$ where w is the R-joint spmf for p and q and $h\ (x, y)$ denotes the R'-joint spmf for $f\ x$ and $g\ y$ for all $x\ R\ y$.

As function application preserves parametricity, any combination of parametric functions is parametric, too. For example, parametricity of $\mathsf{map}_{\mathsf{spmf}}$ and assert follows. Similarly, $\mathsf{try}\ \_\ \mathsf{else}\ \_$ is parametric, too. Thus, this extends to whole probabilistic programs, which we will exploit in §3.4. Such parametricity proofs are highly automated in Isabelle [31,35].

For reasoning about probabilistic programs, we derive more conventional rules by supplying some arguments. For example, we get the following rules for the monad operations. Note that parametricity dictates the shape of the rules.

$$\frac{x\ R\ y}{\mathsf{return}_{\mathsf{spmf}}\ x\ (\mathsf{rel}_{\mathsf{spmf}}\ R)\ \mathsf{return}_{\mathsf{spmf}}\ y}$$

$$\frac{p\ (\mathsf{rel}_{\mathsf{spmf}}\ R)\ q \qquad \forall (x, y) \in R.\ f\ x\ (\mathsf{rel}_{\mathsf{spmf}}\ R')\ g\ y}{\mathsf{bind}_{\mathsf{spmf}}\ p\ f\ (\mathsf{rel}_{\mathsf{spmf}}\ R')\ \mathsf{bind}_{\mathsf{spmf}}\ q\ g}$$

[2] As our embedding is shallow, we cannot define a deduction system in the logic. Rather, we derive the rules directly from the semantics, i.e., show soundness. Completeness is therefore achieved dynamically: new rules can be derived if necessary, in particular when a new operation is introduced.

However, not all functions are parametric. The function $\mathsf{uniform}\ A$, e.g., is not, because it relies on polymorphic equality: the cardinality of a set depends on the equality of elements. In detail, the relator for $\mathsf{rel_{set}}\ R$ relates two sets A and B iff R relates each element of A with one in B and vice versa; formally, $\forall x \in A.\ \exists y \in B.\ x\ R\ y$ and $\forall y \in B.\ \exists x \in A.\ x\ R\ y$. Now,

$$\mathsf{uniform}\ (\mathsf{rel_{set}}\ R \Mapsto \mathsf{rel_{spmf}}\ R)\ \mathsf{uniform} \tag{3}$$

holds if (and only if) the relation R respects equality, i.e., $(=)\ (R \Mapsto R \Mapsto \mathsf{rel_{bool}})\ (=)$ holds, where $\mathsf{rel_{bool}}$ is the identity relation on Booleans (type bool). Interestingly, this restricted parametricity property is equivalent to optimistic sampling in cryptographic proofs. Namely, if f is injective on A, then

$$\mathsf{map_{spmf}}\ f\ (\mathsf{uniform}\ A) = \mathsf{uniform}\ (f\ `\ A) \tag{4}$$

where $f\ `\ A$ denotes the image of A under f. This is one example of Wadler's free theorems [50] in our context. If we specialise A to bitstrings of a given length and f to the bitwise exclusive or (xor) with a fixed bitstring, we obtain the well-known one-time-pad lemma: $\mathsf{map_{spmf}}\ (\mathsf{xor}\ s)\ (\mathsf{uniform}\ \{\,0,1\,\}^n) = \mathsf{uniform}\ \{\,0,1\,\}^n$ where s is a bitstring of length n and $\{\,0,1\,\}^n$ denotes the set of all bitstrings of length n.

Parametricity also connects the relator with probabilities of events. Recall that $\mathsf{measure}\ p\ A$ denotes the probability of event A under the spmf p. The rule

$$\frac{p\ (\mathsf{rel_{spmf}}\ R)\ q \qquad A\ (\mathsf{rel_{pred}}\ R)\ B}{\mathsf{measure}\ p\ A = \mathsf{measure}\ q\ B}$$

follows directly from parametricity of $\mathsf{measure}$, namely

$$\mathsf{measure}\ (\mathsf{rel_{spmf}}\ R \Mapsto \mathsf{rel_{pred}}\ R \Mapsto \mathsf{rel_{real}})\ \mathsf{measure}$$

Here, the relator $\mathsf{rel_{pred}}$ treats sets as predicates, i.e., $\mathsf{rel_{pred}}\ R\ A\ B$ iff $x \in A \leftrightarrow y \in B$ for all $x\ R\ y$, and $\mathsf{rel_{real}}$ is the identity relation on real numbers.

For example, this rule plays an important role in Bellare's and Rogaway's fundamental lemma [16]. Lacking syntax, we cannot express their syntactic condition in HOL. Borrowing ideas from EasyCrypt [13], we instead rephrase the condition in terms of the relator.

Lemma 2 (Fundamental lemma [13,16]). *Let A, F_1 and B, F_2 be events of two spmfs p and q, respectively, such that*

$$p\ (\mathit{rel_{spmf}}\ \{\,(a,b).\ a \in F_1 \leftrightarrow b \in F_2 \wedge (b \notin F_2 \longrightarrow a \in A \leftrightarrow b \in B)\,\})\ q.$$

Then, the probability difference between A occurring in p and B in q is bounded by the probability of F_1 in p, which equals F_2's in q.

$$|\mathit{measure}\ p\ A - \mathit{measure}\ q\ B| \le \mathit{measure}\ p\ F_1 = \mathit{measure}\ q\ F_2$$

Optimistic sampling and the fundamental lemma have illustrated how cryptographic arguments follow from parametricity. But parametricity offers yet another point of view. Mitchell [40] uses parametricity to express representation independence, i.e., one can change the representation of data without affecting the overall result. In §3.4, we will exploit representation independence in the bridging steps of the game transformations.

The Fixpoint Combinator We have not yet covered one important building block of our probabilistic language in our analysis: the fixpoint combinator on spmfs. It turns out that it preserves parametricity.[3]

Theorem 1 (Parametricity of spmf fixpoints).
If $f :: \alpha\ \mathsf{spmf} \Rightarrow \alpha\ \mathsf{spmf}$ and $g :: \beta\ \mathsf{spmf} \Rightarrow \beta\ \mathsf{spmf}$ are monotone w.r.t. $\sqsubseteq$ and $f\ (\mathsf{rel_{spmf}}\ R \Mapsto \mathsf{rel_{spmf}}\ R)\ g$, then $\mathsf{fix}\ f\ (\mathsf{rel_{spmf}}\ R)\ \mathsf{fix}\ g$.

To avoid higher-kinded types, the theorem generalises parametricity to the preservation of relatedness. In the typical use case, f and g are instances of the same polymorphic function.

We prove Thm. 1 by parallel induction on the two fixpoints. Both inductive cases are trivial. The base case requires $\mathsf{rel_{spmf}}\ R$ to be strict, i.e., it relates the least elements, which holds trivially. The step case is precisely the relatedness condition of f and g which the theorem assumes. The hard part consists of showing that parallel induction is a valid proof principle. To that end, we must show that $\mathsf{rel_{spmf}}\ R$ is admissible. A relation $R :: (\alpha \times \beta)\ \mathsf{set}$ is admissible w.r.t. two ccpos iff for any chain $Y \subseteq R$ of pairs in the product ccpo (the ordering is component-wise), R relates the component-wise lubs of Y.

Proposition 2. *$\mathsf{rel_{spmf}}$ is admissible.*

Proof. We exploit the characterisation of $\mathsf{rel_{spmf}}$ in terms of $\mathsf{measure}$. We must show $(\bigsqcup(\pi_1 \mathbin{`} Y))\ (\mathsf{rel_{spmf}}\ R)\ (\bigsqcup(\pi_2 \mathbin{`} Y))$ for all chains Y of pairs in $\mathsf{rel_{spmf}}\ R$. By the characterisation (Lem. 1), this holds by the following reasoning. The first and last step exploit that the lub commutes with $\mathsf{measure}$, and the inequality follows from monotonicity of SUP and the characterisation of $\mathsf{rel_{spmf}}$ for elements of the chain.

$$\begin{aligned}\mathsf{measure}\ (\textstyle\bigsqcup(\pi_1 \mathbin{`} Y))\ A &= \mathsf{SUP}\ (p, \_) \in Y.\ \mathsf{measure}\ p\ A \\ &\le \mathsf{SUP}\ (\_, q) \in Y.\ \mathsf{measure}\ q\ \{y.\ \exists x \in A.\ x\ R\ y\} \\ &= \mathsf{measure}\ (\textstyle\bigsqcup(\pi_2 \mathbin{`} Y))\ \{y.\ \exists x \in A.\ x\ R\ y\}\end{aligned}$$

Note that it is not clear how to prove admissibility via the characterisation in terms of joint spmfs. The issue is that the joint spmfs for the pairs in the chain need not form a chain themselves. So we cannot construct the joint spmf for the lubs as the lub of the joint spmfs.

Admissibility of relators is preserved by the function space (ordered pointwise) and products (ordered component-wise). Thus, analogues to Thm. 1 hold for fixpoints over $\_ \Rightarrow \alpha\ \mathsf{spmf}$, $\alpha\ \mathsf{spmf} \times \beta\ \mathsf{spmf}$, etc. They are useful to show parametricity of (mutually) recursive probabilistic functions (§3.3) rather than recursively defined spmfs.

[3] Wadler showed that if all types are ω-ccpos, all functions are continuous and all relations are admissible and strict, then the fixpoint operator (defined by countable iteration) is parametric [50]. We do not consider the fixpoint operator as part of the language itself, but as a definition principle for recursive user-specified functions. That is, we assume that fix is always applied to a (monotone) function. Consequently, preservation of parametricity suffices and we do not need Wadler's restrictions of the semantic domains. Instead, monotonicity (instead of continuity, see the discussion in §2.3) is expressed as a precondition on the given functions.

2.5 Security of Elgamal Encryption

We are now ready to prove Elgamal encryption (§2.1) IND-CPA secure under the decisional Diffie-Hellman (DDH) assumption. The security theorem bounds the IND-CPA advantage by the DDH advantage. For Elgamal, we prove

$$\mathsf{adv\text{-}ind\text{-}cpa}\ \mathcal{A} = \mathsf{adv\text{-}ddh}\ (\mathsf{elgamal}\ \mathcal{A})$$

where the reduction elgamal transforms an IND-CPA adversary into a DDH adversary as shown in Fig. 3.

The proof consists of two steps. First, observe that ddh_0 ($\mathsf{elgamal}\ \mathcal{A}$) = $\mathsf{ind\text{-}cpa}\ \mathcal{A}$, because after the definitions have been unfolded, both sides are the same except for associativity and commutativity of $\mathsf{bind}_{\mathsf{spmf}}$ and the group law $(\mathsf{g}\,\hat{}\,x)\,\hat{}\,y = \mathsf{g}\,\hat{}\,(x \cdot y)$. Second, we show that ddh_1 ($\mathsf{elgamal}\ \mathcal{A}$) = coin. Note that the message m is independent of γ, which is sampled uniformly. Multiplication with a fixed group element m is a bijection on the carrier. By (4), we can omit the multiplication and the guess b' becomes independent of b. Hence, the adversary has to guess a random bit, which is equivalent to flipping a coin.

```
elgamal $(\mathcal{A}_1, \mathcal{A}_2)\ \alpha\ \beta\ \gamma$ = try do {
  $b \leftarrow$ coin;
  $((m_0, m_1), \sigma) \leftarrow \mathcal{A}_1\ \alpha$;
  assert (valid-plain $m_0 \wedge$ valid-plain $m_1$);
  let $m$ = if $b$ then $m_0$ else $m_1$;
  $b' \leftarrow \mathcal{A}_2\ (\beta, \gamma \otimes m)\ \sigma$;
  $\mathsf{return}_{\mathsf{spmf}}\ (b = b')$
} else coin
```

Fig. 3: Reduction for Elgamal encryption

Formally, the second proof is broken up into three step. First, we rewrite the game using the identities about $\mathsf{bind}_{\mathsf{spmf}}$, try, and $\mathsf{map}_{\mathsf{spmf}}$ such that we can apply (4) on the multiplication. Second, we rewrite the resulting game such that we can apply (4) once more on the equality test. Finally, we show that the irrelevant assignments cancel out. This holds even if the adversary is not lossless thanks to the surrounding try _ else coin. Thus, our statement does not need any technical side condition like losslessness of the adversary in CertiCrypt [14] and EasyCrypt [13].

Modules Note that the definition of IND-CPA security does not depend on the Elgamal encryption scheme. In the formalisation, we abstract over the encryption scheme using Isabelle's module system [11]. Like an ML-style functor, the IND-CPA module takes an encryption scheme and specialises the definitions of game and advantage and the proven theorems to the scheme. Similarly, the DDH assumption has its own module which takes the group as argument. This allows to reuse security definitions and cryptographic algorithms in different contexts. For Elgamal, we import the DDH module for the given group and the IND-CPA module for the Elgamal encryption scheme.

3 Adversaries with Oracle Access

In many security games, the adversary is granted black-box access to oracles. An oracle is a probabilistic function which maintains mutable state across different invocations, but the adversary must not get access to this state. In this section, we propose a new semantic domain for probabilistic functions with oracle access (§3.2). Like in §2.4, we derive reasoning rules from parametricity and explore the

```
aenc $\alpha$ $m$ = do {
  $y \leftarrow$ uniform $\{\,0 ..< |\mathcal{G}|\,\}$;
  $h \leftarrow$ call $(\alpha\ \hat{}\ y)$;
  $\mathsf{return}_{\mathsf{gpv}}$ (xor $h$ $m$) }
```

(a) Hashed Elgamal encryption

```
record $X$ $x$ = do {
  $y \leftarrow$ call $x$;
  $\mathsf{return}_{\mathsf{gpv}}$ $(y$, insert $x$ $X)$ }
```

(b) Recorder for oracle calls

```
ind-cpa $(\mathcal{A}_1, \mathcal{A}_2)$ = try do {
  $(pk, sk) \leftarrow$ key-gen;
  $b \leftarrow$ sample coin;
  $((m_0, m_1), \sigma) \leftarrow \mathcal{A}_1\ pk$
  assert (valid-plain $m_0 \wedge$ valid-plain $m_1$);
  $c^* \leftarrow$ aenc $pk$ (if $b$ then $m_0$ else $m_1$);
  $b' \leftarrow \mathcal{A}_2\ c^*\ \sigma$;
  $\mathsf{return}_{\mathsf{gpv}}$ $(b = b')$
} else sample coin
```

(c) IND-CPA security game with oracle access

Fig. 4: Example programs in the gpv monad

connection to bisimulations (§3.3). We motivate and illustrate the key features by formalising a hashed version of Elgamal encryption (HEG) in the random oracle model (ROM) [15] (§3.1) and verifying its security under the computational Diffie-Hellman (CDH) assumption (§3.4).

3.1 Example: Hashed Elgamal Encryption

Elgamal's encryption scheme from §2.1 requires messages to be group elements, but often bitstrings of a given length are more convenient. Therefore, we consider a version of Elgamal encryption where a hash function H converts group elements into bitstring [48]. We model the hash function as a random oracle, which acts like a random function. In detail, we replace the multiplication with the group element $\alpha\ \hat{}\ y$ in Fig. 1a with the xor of its hash h; Fig. 4a shows the resulting encryption algorithm. The encryption algorithm obtains the hash by calling the oracle with a random group element.

In §3.4, we prove that HEG is IND-CPA secure under the computational Diffie Hellman (CDH) assumption. For now, we explain how the formalisation changes. Figure 4c shows the new game for chosen plaintext attacks where the key generation algorithm, the encryption function and the adversary have access to an oracle. In comparison to Fig. 1c, the monad has changed: the game uses the new monad of generative probabilistic values (gpv) rather than spmf; see §3.2 for details. Accordingly, the coin flips coin are embedded with the monad homomorphism sample.

In one step of the security proof, we will have to keep track of the calls that the adversary has made. This is achieved by using the oracle transformation record in Fig. 4b. It forwards all calls x and records them in its local state X.

As before, all these definitions live in different modules that abstract over the encryption scheme, group, and oracle. In fact, the programs in Fig. 4 are completely independent of the concrete oracle. We compose the game with the oracle only when we define the advantage. Thus, it now depends on an oracle $\mathcal{O}$ with initial state s: adv-ind-cpa $\mathcal{O}$ s $\mathcal{A}$ = |run $\mathcal{O}$ (ind-cpa $\mathcal{A}$) s ! True $- 1/2$|. Here, run binds the oracle calls in ind-cpa $\mathcal{A}$ to the oracle $\mathcal{O}$. It thus reduces a program with oracle access to one without, i.e., a spmf.

3.2 Generative Probabilistic Values

The example from the previous section determines a wish list of features for the language of probabilistic programs with oracle access: assertions and failure handling, calls to unspecified oracles, embedding of probabilistic programs without oracle access, and composition operators. In this section, we propose a semantic domain for probabilistic computations with oracle access and show how to express the above features in this domain.

We start by discussing why spmfs do not suffice. In our probabilistic language of spmfs, we can model an oracle as a function of type $\sigma \Rightarrow \alpha \Rightarrow (\beta \times \sigma)\ \mathsf{spmf}$, which takes a state and the arguments of the call and returns a subprobability distribution over the output and the new state. Unfortunately, we cannot model the adversary as a probabilistic program parametrised over the oracle and its initial state. Suppose we do. Then, its type has the shape $(\sigma{\Rightarrow}\alpha{\Rightarrow}(\beta{\times}\sigma)\ \mathsf{spmf}){\Rightarrow}$ $\sigma \Rightarrow \_\ \mathsf{spmf}$. To hide the oracle state from the adversary despite passing it as an argument, we could require that the adversary be parametric in σ. This expresses that the adversary behaves uniformly for all states, so it cannot inspect the state. Yet, we must also ensure that the adversary uses the state only linearly, i.e., it does not call the oracle twice with the same state. As we are not aware of a semantic characterisation of linearity, we cannot model the adversary like this.

Instead, we explicitly model the interactions between the adversary and the oracle. To that end, we propose the following algebraic codatatype of *generative probabilistic values* (gpv):[4]

$$\begin{array}{l}\textbf{codatatype}\ (\alpha,\gamma,\rho)\ \mathsf{gpv} = \mathsf{GPV}\ (\alpha + \gamma \times (\alpha,\gamma,\rho)\ \mathsf{rpv}))\ \mathsf{spmf}\\ \textbf{type-synonym}\ (\alpha,\gamma,\rho)\ \mathsf{rpv} = \rho \Rightarrow (\alpha,\gamma,\rho)\ \mathsf{gpv}\end{array}$$

Conceptually, a gpv is a probabilistic system in which each state chooses probabilistically between failing, terminating with a result of type α, and continuing by producing an output γ and transitioning into a *reactive probabilistic value* (rpv), which waits for a non-deterministic response of the environment (e.g., the oracle) of type ρ. Upon reception, the rpv transitions to the generative successor state.

As we are interested in a shallow embedding, the type gpv only models the observations (termination with result or failure and interaction with the environment) of the system rather than the whole system with the states. This yields a richer equational theory, i.e., we can prove more properties by rewriting without resorting to bisimulation arguments. Any probabilistic system with explicit states can be transformed into a gpv by identifying bisimilar states. The coiterator $\mathsf{coiter}_{\mathsf{gpv}} :: (\sigma{\Rightarrow}(\alpha{+}\gamma{\times}(\rho{\Rightarrow}\sigma))\ \mathsf{spmf}){\Rightarrow}\sigma{\Rightarrow}(\alpha,\gamma,\rho)\ \mathsf{gpv}$ formalises this: given a probabilistic system and an initial state, it constructs the corresponding gpv.

Basic operations The basic operations for gpv are the monadic functions $\mathsf{return}_{\mathsf{gpv}}$ and $\mathsf{bind}_{\mathsf{gpv}}$, calling an oracle call, sampling sample, exceptional termination fail (from which we derive assertions assert) and failure handling try _ else _. They

[4] Note that gpv is not the greatest fixpoint of a polynomial functor, as the recursion goes through the non-polynomial functor spmf. Still, the type is well-defined, as spmf is a bounded natural functor [30] which Isabelle's codatatype package supports [22].

$$\begin{array}{ll}
\mathsf{return_{gpv}}\ x & = \mathsf{GPV}\ (\mathsf{return_{spmf}}\ (\mathsf{Pure}\ x)) \\
\mathsf{call}\ o & = \mathsf{GPV}\ (\mathsf{return_{spmf}}\ (\mathsf{IO}\ o\ \mathsf{return_{gpv}})) \\
\mathsf{sample}\ p & = \mathsf{GPV}\ (\mathsf{map_{spmf}}\ \mathsf{Pure}\ p) \\
\mathsf{fail} & = \mathsf{GPV}\ \bot \\
\mathsf{assert}\ b & = \mathsf{if}\ b\ \mathsf{then}\ \mathsf{return_{gpv}}\ ()\ \mathsf{else}\ \mathsf{fail} \\
\multicolumn{2}{l}{\mathsf{try}\ \mathsf{GPV}\ p\ \mathsf{else}\ v =} \\
\multicolumn{2}{l}{\quad \mathsf{GPV}\ (\mathsf{try}\ \mathsf{map_{spmf}}\ (\mathsf{map_{+}}\ \mathsf{id}\ (\mathsf{map_{\times}}\ \mathsf{id}\ (\lambda r\ x.\ \mathsf{try}\ (r\ x)\ \mathsf{else}\ v)))\ p\ \mathsf{else}\ \mathsf{un\text{-}GPV}\ v)}
\end{array}$$

$$\begin{array}{l}
\mathsf{bind_{gpv}}\ (\mathsf{GPV}\ p)\ f = \mathsf{GPV}\ (\mathbf{do}\ \{ \\
\quad x \leftarrow p; \\
\quad \mathsf{case}\ x\ \mathsf{of}\ \mathsf{Pure}\ y \Rightarrow \mathsf{un\text{-}GPV}\ (f\ y) \\
\quad \mid \mathsf{IO}\ c\ r \Rightarrow \mathsf{return_{spmf}} \\
\qquad (\mathsf{IO}\ c\ (\lambda w.\ \mathsf{bind_{gpv}}\ (r\ w)\ f))\ \})
\end{array}$$

Fig. 5: Primitive operations for gpvs

can be implemented as shown in Fig. 5, where $\mathsf{Pure} = \mathsf{Inl}$ and $\mathsf{IO}\ o\ r = \mathsf{Inr}\ (o, r)$ and id is the identity and un-GPV the inverse of GPV. Note that $\mathsf{bind_{spmf}}$ and try _ else _ can be defined by primitive corecursion.

The operations behave as expected. In particular, $\mathsf{return_{gpv}}$ and $\mathsf{bind_{gpv}}$ satisfy the monad laws, fail propagates, and sample is a monad homomorphism.

$$\mathsf{bind_{gpv}}\ \mathsf{fail}\ f = \mathsf{fail} \qquad \mathsf{sample}\ (\mathsf{return_{spmf}}\ x) = \mathsf{return_{gpv}}\ x \qquad \mathsf{sample}\ \bot = \mathsf{fail}$$

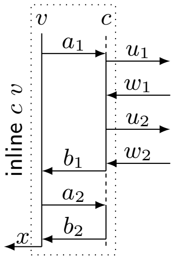
$$\mathsf{sample}\ (\mathbf{do}\ \{x \leftarrow p;\ f\ x\ \}) = \mathbf{do}\ \{\ x \leftarrow \mathsf{sample}\ p;\ \mathsf{sample}\ (f\ x)\ \}$$

$$\mathsf{sample}\ (\mathsf{assert}\ b) = \mathsf{assert}\ b \qquad \mathsf{sample}\ (\mathsf{try}\ p\ \mathsf{else}\ q) = \mathsf{try}\ \mathsf{sample}\ p\ \mathsf{else}\ \mathsf{sample}\ q$$

The resulting equational theory is rich again, but not as rich as for spmfs. Commutativity (1) and cancellation (2), e.g, do not carry over to $\mathsf{bind_{gpv}}$ in general.

Composition operators Two gpvs can be composed such that one (the callee) processes all the calls of the other (the caller). Thereby, the callee may issue further calls. In games, composition is mainly used to intercept and redirect the oracle calls of the adversary, i.e., the callee is an oracle transformation like record. Syntactically, composition corresponds to inlining the callee into the caller. If programs were modelled syntactically, implementing inlining would be trivial; but with a shallow approach, we cannot rely on syntax.

Instead, we define inlining by a combination of recursion and corecursion as shown in Fig. 6. The sequence diagram on the right illustrates what is happening in an example. The caller v issues calls of type γ which return values of type ρ. The callee c is a function from γ to a gpv which may issue further calls of type γ' which return values of type ρ'. The callee maintains its own state of type σ between invocations. Therefore, like oracles, the callee additionally takes a state as argument and the results of the gpv are the return values and the new states. The function inline first calls the auxiliary function search, which searches for the first call issued by the callee during a call by the caller. If search finds none, it returns the result x of the caller and the updated state s' of the callee. Then, inline terminates with the same outcome. Otherwise, inline issues the call u and forwards the return value w to the rpv r' of the callee, which may issue further calls (u_2 in the diagram). The result b of the callee is then fed to the rpv r of the caller and inline corecurses with the updated state s' of the callee.



```
inline :: $(\sigma \Rightarrow \gamma \Rightarrow (\rho \times \sigma, \gamma', \rho')$ gpv)
  $\Rightarrow (\alpha, \gamma, \rho)$ gpv $\Rightarrow \sigma$
  $\Rightarrow (\alpha \times \sigma, \gamma', \rho')$ gpv

inline $c\ v\ s$ = GPV (do {
   $z \leftarrow$ search $c\ v\ s$
   case $z$ of Inl $(x, s') \Rightarrow$ Pure $(x, s')$
    | Inr $(u, r', r) \Rightarrow$
       IO $u$ ($\lambda w$. do {
          $(b, s') \leftarrow r'\ w$;
          inline $c\ (r\ b)\ s'$ } })
```

```
search :: $(\sigma \Rightarrow \gamma \Rightarrow (\rho \times \sigma, \gamma', \rho')$ gpv)
  $\Rightarrow (\alpha, \gamma, \rho)$ gpv $\Rightarrow \sigma$
  $\Rightarrow (\alpha \times \sigma +$
       $\gamma' \times (\rho \times \sigma, \gamma', \rho')$ rpv $\times (\alpha, \gamma, \rho)$ rpv) spmf

search $c\ v\ s$ = do {
   $z \leftarrow$ un-GPV $v$;
   case $z$ of Pure $x \Rightarrow$ return$_{\mathsf{spmf}}$ (Inl $(x, s)$)
    | IO $a\ r \Rightarrow$ do {
       $y \leftarrow$ un-GPV $(c\ s\ a)$;
       case $y$ of Pure $(b, s') \Rightarrow$ search $c\ (r\ b)\ s'$
        | IO $u\ r' \Rightarrow$ return$_{\mathsf{spmf}}$ (Inr $(u, r', r)$) } }
```

Fig. 6: Composition operator for gpvs

The function search recursively goes through the interactions between the caller and the callee. If the caller terminates with result x, there are no calls and the search terminates. Otherwise, the caller issues a call a and becomes the rpv r. In that case, search analyses the callee under the argument a. If the callee returns b without issuing a call itself, the search continues recursively on $r\ b$. Otherwise, the first call is found and the search terminates.

Note that search operates in the spmf monad. So, it can be defined using the fixpoint operator on spmf (§2.3). Conversely, inline operates in the gpv monad. So, corecursion is the appropriate definition principle. Accordingly, we prove properties about search by fixpoint induction and about inline by coinduction. For example, we show that inline is a monad homomorphism. It is also associative (we omit reassociations of tuples for clarity; $f \circ\!\circ g$ denotes $\lambda(x, y)\ z.\ f\ (g\ x\ z)\ y$):

$$\mathsf{inline}\ c_1\ (\mathsf{inline}\ c_2\ v\ s_2)\ s_1 = \mathsf{inline}\ (\mathsf{inline}\ c_1 \circ\!\circ c_2)\ v\ (s_2, s_1)$$

If the callee is an oracle $\mathcal{O}$, i.e., an spmf rather than a gpv, it cannot issue further calls. Thus, search $\mathcal{O}$ always returns a result of the form Inl (x, s'), i.e., the corecursion in inline is not needed. Therefore, we define the execution of a gpv v with $\mathcal{O}$ as follows (where projl is the left inverse to Inl).

$$\mathsf{exec} :: (\sigma \Rightarrow \gamma \Rightarrow \rho \times \sigma\ \mathsf{spmf}) \Rightarrow (\alpha, \gamma, \rho)\ \mathsf{gpv} \Rightarrow \sigma \Rightarrow (\alpha \times \sigma)\ \mathsf{spmf}$$
$$\mathsf{exec}\ \mathcal{O}\ v\ s = \mathsf{map}_{\mathsf{spmf}}\ \mathsf{projl}\ (\mathsf{search}\ (\lambda s\ x.\ \mathsf{sample}\ (\mathcal{O}\ s\ x))\ v\ s)$$

When $\mathcal{O}$'s final state does not matter, we use run $c\ v\ s$ = map$_{\mathsf{spmf}}$ π_1 (exec $c\ v\ s$).

Reductions are the primary use case for composition. They transform the adversary $\mathcal{A}$ for one game into an adversary for another game. In general, the oracles of the two games differ. So, the reduction emulates the original oracle $\mathcal{O}$ using an oracle transformation T, which has access to the new oracle $\mathcal{O}'$. In this view, the new adversary is built from the composition inline T $\mathcal{A}$ and the cryptographic assumption executes it with access to $\mathcal{O}'$. By associativity of composition (see Fig. 7), this is equivalent to executing the original adversary $\mathcal{A}$ with access to the emulated oracle exec $\mathcal{O}' \circ\!\circ T$. Thus, it suffices to establish that the emulation exec $\mathcal{O}' \circ\!\circ T$ of $\mathcal{O}$ is good enough.

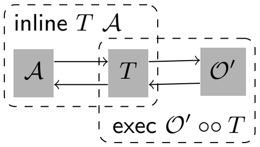


Fig. 7: Associativity of composition illustrated

3.3 Parametricity and Bisimulation

Our framework provides a set of rules (a logic) for reasoning about the relation between games with oracles. This logic complements the equational theory derived from the shallow embedding. Like for spmf, parametricity guides us in choosing the rules and proving them sound. As the first step, we therefore define a relator $\textsf{rel}_{\textsf{gpv}}$ for gpvs. As gpvs form a codatatype, the canonical definition is as a coinductive relation, namely the one specified by (5).[5]

$$\frac{\textsf{un-GPV}\ v\ (\textsf{rel}_{\textsf{spmf}}\ (\textsf{rel}_{+}\ A\ (\textsf{rel}_{\times}\ C\ ((=) \Mapsto \textsf{rel}_{\textsf{gpv}}\ A\ C))))\ \textsf{un-GPV}\ v'}{v\ (\textsf{rel}_{\textsf{gpv}}\ A\ C)\ v'} \tag{5}$$

Canonicity ensures parametricity of the coiterator $\textsf{coiter}_{\textsf{gpv}}$, the constructor $\textsf{GPV}$, and the selector $\textsf{un-GPV}$. For $\textsf{coiter}_{\textsf{gpv}}$, e.g., we obtain that for all S, A, and C,

$$\textsf{coiter}_{\textsf{gpv}}\ ((S \Mapsto \textsf{rel}_{\textsf{spmf}}\ (\textsf{rel}_{+}\ A\ (\textsf{rel}_{\times}\ C\ ((=) \Mapsto S)))) \Mapsto S \Mapsto \textsf{rel}_{\textsf{gpv}}\ A\ C)\ \textsf{coiter}_{\textsf{gpv}}$$

Consequently, all the primitive operations in Fig. 5 are parametric, too. As the fixpoint operator preserves parametricity (Thm. 1), $\textsf{search}$ is also parametric. And so are $\textsf{inline}$ and $\textsf{exec}$ and $\textsf{run}$. Thus, parametricity links $\textsf{rel}_{\textsf{gpv}}$ with all the gpv operations.

Similar to spmfs (§2.4), this link leads to rules for reasoning about game transformations. We do not go into the details here. Instead, we consider the example of replacing an oracle with a bisimilar one as formalised by (6). This rule follows from the parametricity of $\textsf{run}$ by unfolding the definitions.[6] The premises express that S is a bisimulation relation between the oracles $\mathcal{O}_1$ and $\mathcal{O}_2$ and relates their initial states. Bisimulation means that whenever two states s_1 and s_2 are related and the oracles are called with c, then they return the same value and the states are related again. The "then" part is expressed by the relation $\textsf{rel}_{\times}\ (=)\ S$ which $\textsf{rel}_{\textsf{spmf}}$ lifts to subprobability distributions. The second premise states that S relates the initial states. In the conclusion, we get that running a gpv v (e.g., the adversary) with two bisimilar oracles produces the same outcomes.

$$\frac{\forall s_1\ s_2\ c.\ s_1\ S\ s_2 \longrightarrow \mathcal{O}_1\ s_1\ c\ (\textsf{rel}_{\textsf{spmf}}\ (\textsf{rel}_{\times}\ (=)\ S))\ \mathcal{O}_2\ s_2\ c \qquad s_1\ S\ s_2}{\textsf{run}\ \mathcal{O}_1\ v\ s_1 = \textsf{run}\ \mathcal{O}_2\ v\ s_2} \tag{6}$$

[5] The type constructor $\textsf{gpv}$ takes three type arguments, so we should expect $\textsf{rel}_{\textsf{gpv}}$ to take three relations, too. However, the third argument occurs in a negative position in the codatatype definition. Therefore, the relator does not enjoy useful properties such as monotonicity and distributivity over relation composition. Consequently, Isabelle's infrastructure for parametricity treats these arguments as fixed (dead in the terminology of [22]). So, $\textsf{rel}_{\textsf{gpv}}$ takes only relations for the first two arguments and fixes the third to the identity relation $(=)$ as can be seen in (5). In practice, we have not found this specialisation to be restrictive.

[6] In fact, parametricity actually yields a rule stronger than (6), namely the gpv v need not be the same on both sides. If $v_1\ (\textsf{rel}_{\textsf{gpv}}\ A\ (=))\ v_2$ and the premises of (6) hold, then $\textsf{run}\ \mathcal{O}_1\ v_1\ s_1\ (\textsf{rel}_{\textsf{spmf}}\ A)\ \textsf{run}\ \mathcal{O}_2\ v_2\ s_2$.

```
lcdh $\mathcal{A}$ = do {
  $x \leftarrow$ uniform $\{0 ..< |\mathcal{G}|\}$;
  $y \leftarrow$ uniform $\{0 ..< |\mathcal{G}|\}$;
  $Z \leftarrow \mathcal{A}$ (g ^ $x$) (g ^ $y$);
  return$_{\mathsf{spmf}}$ (g ^ $(x \cdot y) \in Z$) }
```

Fig. 8: Computational Diffie-Hellman game

```
hash-$\mathcal{O}$ $s_{\mathcal{O}}$ $x$ =
  case $s_{\mathcal{O}}$ $x$ of None $\Rightarrow$ do {
    $bs \leftarrow$ uniform $\{0, 1\}^n$;
    return$_{\mathsf{spmf}}$ $(bs, s_{\mathcal{O}}(x \mapsto bs))$ }
  | Some $bs \Rightarrow$ return$_{\mathsf{spmf}}$ $(bs, s_{\mathcal{O}})$
```

Fig. 9: Random hash oracle

In fact, the derivation may be seen as an instance of representation independence [40,44]. By design, the gpv has only black-box access to the oracle, i.e., they interface only via calls and returns. Thus, the first premise expresses exactly the requirement of representation independence: a relation S that is preserved by all operations of the interface. In Isabelle [31], so-called transfer rules express preservation in point-free style; here, $\mathcal{O}_1$ $(S \Rrightarrow (=) \Rrightarrow \mathsf{rel}_{\mathsf{spmf}}\ (\mathsf{rel}_{\times}\ (=)\ S))$ $\mathcal{O}_2$. The HEG example exploits this idea in two game transformations (§3.4).

There are also limits to what we can derive from parametricity. For example, consider the case where only one of the oracles enters an error state, say because the adversary has guessed a secret. Then, the adversary can distinguish between the oracles and behave differently, i.e., it is not related by $\mathsf{rel}_{\mathsf{gpv}}$ any more. Therefore, we would need a notion of bisimulation up to error, but relational parametricity cannot express this. Thus, we prove the Lem. 3 directly.

Lemma 3 (Oracle bisimulation up to error states). *Let S be a relation between the states of two oracles $\mathcal{O}_1$ and $\mathcal{O}_2$ and let F_1 and F_2 be the sets of error states. Define the relation R by (x, s_1) R (y, s_2) iff $s_1 \in F_1 \leftrightarrow s_2 \in F_2$ and if $s_2 \notin F_2$ then $x = y$ and s_1 S s_2. Then exec $\mathcal{O}_1$ v s_1 ($\mathit{rel}_{\mathit{spmf}}$ R) exec $\mathcal{O}_2$ v s_2 if*

- *$\mathcal{O}_1$ $(S \Rrightarrow (=) \Rrightarrow \mathit{rel}_{\mathit{spmf}}\ R)$ $\mathcal{O}_2$, and*
- *s_1 S s_2 and $s_1 \in F_1 \leftrightarrow s_2 \in F_2$, and*
- *$\mathcal{O}_1$ and $\mathcal{O}_2$ are lossless in error states and error states are never left, and*
- *v issues finitely many calls and never fails, i.e., all spmfs in v are lossless.*

3.4 Security of Hashed Elgamal Encryption

Now, we illustrate how our framework supports proofs about games with oracles by reducing the IND-CPA security of Hashed Elgamal encryption to the computational Diffie-Hellman (CDH) assumption. The CDH assumption states that it is hard to compute g ^ $(x \cdot y)$ given g ^ x and g ^ y. For comparison with the existing proofs in Certicrypt [14] and EasyCrypt [13] (§4), we reduce the security to the set version LCDH of CDH, where the adversary may return a finite set of candidates for g ^ $(x \cdot y)$. The reduction from LCDH to CDH is straightforward.

LCDH is formalised as follows. The LCDH adversary $\mathcal{A}$ is a probabilistic function from two group elements to a set of group elements. Its advantage adv-lcdh $\mathcal{A}$ is the probability of True in the security game lcdh shown in Fig. 8.

In the random oracle model [15], hash functions are idealised as random functions. The random oracle hash-$\mathcal{O}$ shown in Fig. 9 implements such a random function. Its state $s_{\mathcal{O}}$ is a map from group elements (inputs) to bitstrings (outputs), which associates previous queries with the outputs. Upon each call, the oracle looks up the input x in the map $s_{\mathcal{O}}$ and returns the corresponding output if found. If not, it returns a fresh bitstring bs of length n and updates the map $s_{\mathcal{O}}$ to associate x with bs.

The security of HEG is shown by a sequence of game transformations. They closely follow [13], so we do not present them in detail here. Instead, we focus on some steps to highlight the use of parametricity, equational reasoning, and the program logic. In the following, we assume a fixed IND-CPA adversary $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$ which makes only finitely many calls and does not fail. This technical restriction is necessary to apply Lem. 3.

The first step changes the game such that it records the adversary's queries to the hash oracle using the oracle transformation record from Fig. 4b. So the first game game_1 is the same as $\mathsf{ind\text{-}cpa}$ from Fig. 4c except that the calls to the adversary are replaced by $(((m_0, m_1), \sigma), X) \leftarrow \mathsf{inline\ record}\ (\mathcal{A}_1\ pk)\ \{\}$ and $(b', X') \leftarrow \mathsf{inline\ record}\ (\mathcal{A}_2\ c^*\ \sigma)\ X$. The equality $\mathsf{ind\text{-}cpa}\ \mathcal{A} = \mathsf{game}_1$ follows from parametricity and bisimilarity of record and the identity oracle transformation $\mathsf{ID} = \lambda \sigma\ x.\ \mathbf{do}\ \{\ y \leftarrow \mathsf{call}\ x;\ \mathsf{return}_{\mathsf{gpv}}\ (y, \sigma)\ \}$ due to representation independence, and the fact that ID is the identity for gpv composition. The bisimulation relation $S = \{\ (\sigma, X).\ \mathsf{True}\ \}$ identifies all states. The equality is proven automatically by Isabelle's prover for representation independence [31] using the transfer rules $()\ S\ \{\}$ and $\mathsf{ID}\ (S \Mapsto (=) \Mapsto \mathsf{rel}_{\mathsf{spmf}}\ (\mathsf{rel}_{\times}\ (=)\ S))\ \mathsf{record}$ and equational rewriting. In fact, the latter rule is another instance of representation independence, as the two oracle transformations differ only in the update of the local state. The transfer rule $(\lambda\_\ s.\ s)\ ((=) \Mapsto S \Mapsto S)\ \mathsf{insert}$ formalises the connection. The parametricity prover derives the transfer rule for the oracles from this.

In the second step, we push the oracle transformation into the oracle exploiting associativity (cf. Fig. 7) and distribute the outermost run to the calls of the adversary. This changes the monad from gpv to spmf, in which $\mathsf{bind}_{\mathsf{spmf}}$ commutes (1). Additionally, the new game game_2 returns a second boolean to flag whether the adversary caused a call to the hash oracle with the group element $\mathsf{g}\,\hat{}\,(x \cdot y)$. By equational reasoning only, we prove that $\mathsf{run\ hash\text{-}}\mathcal{O}\ \mathsf{game}_1\ \varnothing = \mathsf{map}_{\mathsf{spmf}}\ \pi_1\ \mathsf{game}_2$ where the hash oracle starts in the initial state $\varnothing$.

The third transformation replaces the hash for the challenge ciphertext with a random bitstring which is not recorded in the hash oracle's map. By the fundamental lemma, the difference in the success probabilities is bounded by the error event, which is in this case the flag introduced in game_2. The assumption of the fundamental lemma is proven using the rules of the program logic. In particular, we apply Lem. 3 on the call to $\mathcal{A}_2$, as the oracles are not bisimilar if any of the remaining queries calls the hash function on $\mathsf{g}\ \hat{}\ (x \cdot y)$.

```
hElgamal $(\mathcal{A}_1, \mathcal{A}_2)\ \alpha\ \beta$ = do {
  $(((m_0, m_1), \sigma), s) \leftarrow$ exec hash-$\mathcal{O}$ $(\mathcal{A}_1\ \alpha)\ \varnothing$;
  try do {
    assert (valid-plain $m_0 \wedge$ valid-plain $m_1$);
    $h \leftarrow$ uniform $\{0,1\}^n$;
    $(b', s') \leftarrow$ exec hash-$\mathcal{O}$ $(\mathcal{A}_2\ (\beta, h)\ \sigma)\ s$;
    return$_{\mathsf{spmf}}$ (dom $s'$)
  } else return$_{\mathsf{spmf}}$ (dom $s$) }
```

Fig. 10: Reduction for Hashed Elgamal

We do not go into the details of the remainder of the proof, as it applies the same techniques as before. The one time pad lemma is used to make the challenge ciphertext independent of the challenge bit b similar to optimistic sampling in §2.5. As the set of queries now is just the domain of the map of the hash oracle, we replace $\mathsf{exec}\ \mathsf{hash}\text{-}\mathcal{O} \circ\!\circ\, \mathsf{record}$ with $\mathsf{hash}\text{-}\mathcal{O}$ using representation independence. Finally, we show by equational reasoning that the first boolean of the game is just a coin flip and the second corresponds to the LCDH game for the transformed adversary $\mathsf{hElgamal}\ \mathcal{A}$ (see Fig. 10).[7] In summary, we obtain

$$\mathsf{adv\text{-}ind\text{-}cpa}\ \mathsf{hash}\text{-}\mathcal{O}\ \varnothing\ \mathcal{A} \le \mathsf{adv\text{-}lcdh}\ (\mathsf{hElgamal}\ \mathcal{A})$$

4 Comparison with Existing Frameworks

In this section, we compare our framework with the existing tools CertiCrypt [14], EasyCrypt [13], FCF [42], and Verypto [8] in four respects: readability, expressiveness, the trusted codebase, and the formalisation effort.

Readability is important for two reasons. First, security definitions should resemble those in the cryptographic literature such that cryptographers are able to understand and evaluate them. Second, intuitive syntax supports the users during the formalisation, as they can focus on formalising the arguments rather than trying to understand hardly readable programs. All frameworks achieve good readability, but in different ways. CertiCrypt and EasyCrypt embed an imperative procedural language in their logics. They closely model Bellare's and Rogaway's idea of a stateful language with oracles as procedures [16]. Verypto deeply embeds a higher order language with mutable references based on de Bruijn indices in HOL. Readability is regained by reflecting the syntax in HOL's term language using parsing and pretty-printing tricks. In contrast, FCF and our framework shallowly embed a functional language with monadic sequencing in Coq and Isabelle/HOL, respectively. Like in Shoup's treatment [48], the state of the adversary and the oracles must be passed explicitly. This improves clarity as it makes explicit which party can access which parts of the state. Conversely, it can also be a source for errors, as it is the user who must ensure that the states be used linearly.

Expressiveness has two dimensions. First, the syntax and the semantic domain determine the expressiveness of the language. CertiCrypt and EasyCrypt support discrete subprobability distributions and a procedural while language, but no fixpoint operator, although fixpoints could be defined in CertiCrypt's semantic domain similar to our work (§2.3 and Thm. 1). Fixpoints do not increase the expressiveness over a while combinator, but lead to more natural formulations of programs. EasyCrypt additionally provides a module system to support abstraction and reuse. Verypto is the most general framework, as it builds on

[7] The oracle transformation in this reduction is degenerate, as $\mathsf{hElgamal}$ emulates the oracle without access to further oracles. Thus, we use exec directly instead of inline.

measure theory and therefore supports continuous distributions and higher order functions—at the price of incurring measurability proof obligations. FCF's semantics allows only probability distributions with finite support, so no fixpoint operator can be defined. The syntax further restricts probabilistic effects to random sampling of bitstrings and conditional probabilities. For programs with oracle access, FCF provides equivalents to our operations call, sample, and inline. Further effectful operators (such as try _ else _ cannot be added by the user, but require changes to the language formalisation itself. Conversely, FCF's deterministic pure sublanguage includes all functions and types from the library of the proof assistant thanks to the shallow embedding. Abstraction and reuse come from Coq's module system. Like FCF, our framework is integrated with the libraries and facilities of the proof assistant. It extends this advantage to probabilistic programs with oracle access, as we define also these languages directly in the semantic domain. Thus, users can define new control structures on the spot, if needed. It supports discrete subprobabilities similar to CertiCrypt and EasyCrypt and in addition features a fixpoint operator.

Second, the embedding and the logic determine what kind of security properties can be formalised. All frameworks support concrete security proofs. Thanks to the deep embedding, CertiCrypt and Verypto also support statements about efficiency and thus asymptotic security statements, which is impossible in EasyCrypt and ours, because HOL functions are extensional. FCF comes with an axiomatic cost model for efficiency, which is not formally connected to an operational model. This is possible despite the shallow embedding as long as extensionality is not assumed. In these three frameworks, asymptotic bounds are derived from concrete bounds. So no work is saved by reasoning asymptotically.

The trusted code base influences the trustworthiness of a mechanised proof and should be as small as possible. Proof assistants like Coq and Isabelle consist of a small kernel and can additionally produce proof terms or proof objects that an independent checker can certify. Consequently, CertiCrypt, FCF, and our framework achieve high ranks there. Verypto falls behind because its measure theory is only axiomatized rather than constructed definitionally. EasyCrypt does not have a small kernel, so the whole implementation in OCaml and the external SMT solvers must be trusted.

The formalisation effort determines the usability of a framework. For this comparison, we estimate the effort by the proof length measured in line counts. Clearly, proof styles affect line counts, so the numbers must be taken with a grain of salt. Nevertheless, they roughly indicate the effort required to produce such proofs. Table 1 lists the length (in lines) of the security statement for different cryptographic algorithms and frameworks. The line count includes the statement of the concrete security theorem, its proof, and all intermediate games. It does not cover the cryptographic algorithm itself nor the security definition. We obtained the numbers by inspecting the proof scripts distributed with the frameworks. Unfortunately, there are no line counts for Verypto because we could not get access to the sources.

Encryption	Framework			
algorithm	ours	CertiCrypt	EasyCrypt[a]	FCF[b]
Elgamal in the standard model (§2.1)	52	238	58	156
Hashed Elgamal in the ROM (§3.1)	236	810	210	
Pseudo-random function [42]	352			1166

[a] Version 27a6617 on github.com/EasyCrypt/easycrypt.git, 4 Jan 2016
[b] Version a445b73 on github.com/adampetcher/fcf, 13 Dec 2015

Table 1: Framework comparison by line counts of concrete security theorems

As Petcher and Morrisett have already observed [43], shallow embeddings (FCF, ours) have an advantage over deep ones (CertiCrypt, Verypto), as all the reasoning infrastructure and libraries of the proof assistant can be reused directly; a deep embedding would need to encode the libraries in the syntax of the language. Despite being more general, our framework leads to shorter proofs than FCF. We see two reasons for this gap. First, our language works directly in the semantic domain, even for effectful programs. This gives a richer equational theory and all conversions between syntax and semantics become superfluous. For example, their rule for loop fission holds only in the relational logic, but it is a HOL equality in our model. Second, Isabelle's built-in proof automation, in particularly conditional rewriting, provides a reasonable level of automation, especially for the equational proofs mentioned above. So far, we have not yet implemented any problem-specific proof tactics. Such tactics could automate the proofs even more, especially the manual reasoning with commutativity of sequencing. In comparison to EasyCrypt, the state of the art in proof automation, we achieve a similar degree of automation.

5 Further Related Work

In §4, we have already compared our framework with the existing ones for mechanising game playing proofs. In this section, we review further related work.

The tool CryptoVerif by Blanchet [20] can prove secrecy and correspondence properties such as authentication of security protocols. As it can even discover intermediate games itself, the tool achieves a much higher degree of automation than any of the frameworks including ours. The language—a process calculus inspired by the π calculus—distinguishes between a unique output process and possibly many input processes, which communicate via channels. Our gpvs also distinguish between inputs and outputs, but composition works differently. In CryptoVerif, several input processes may be able to receive an output and the semantics picks one uniformly randomly. In our setting, the callee represents all input processes and receives all calls. In principle, one could embed Blanchet's calculus in our semantic domain of gpvs using a different composition operator. Then, CryptoVerif's abstractions could be proven sound in our framework.

The functional programming language F* [17,49] has been used to verify implementations of cryptographic algorithms and protocols [12]. Security properties are formulated as type safety of an annotated, dependently-typed program and the type checker ensures type safety. While this approach scales to larger applications [18], the security properties cannot be stated concisely as the typing assertions are scattered over the whole implementation.

Affeldt et al. [2] formalise pmfs in Coq and apply this to coding theory.

Audebaud and Paulin-Mohring [5] formalised the spmf monad in Coq. They also define the approximation order $\sqsubseteq$ on spmfs and show that it forms an ω-complete partial order, i.e., *countable* chains have least upper bounds. Using Kleene's fixpoint theorem, they obtain a fixpoint operator for continuous functions. We generalise their result in that *arbitrary* spmf chains have least upper bounds. Thus, monotonicity (rather than continuity) suffices for the fixpoints.

CertiCrypt [14] uses their monad as the semantic domain for programs and adds the lifting operator $\mathsf{rel}_{\mathsf{spmf}}$. Zanella Béguelin proves a special case of Thm. 1, where the functions f and g are projections of a joint continuous function [51].

Cock [24] develops a shallow embedding of a probabilistic guarded command language in HOL. Programs are interpreted as monotone transformers of bounded expectations, which form a complete lattice. So, recursive functions can be defined using fixpoints. He focuses on proving functional correctness properties using non-relational Hoare triples and verification conditions.

Our framework reuses the existing infrastructure for relational parametricity in Isabelle/HOL, but in new ways. The Lifting package [31] exploits representation independence to transfer theorems between raw types and their quotients or subtypes. Lammich's tool AutoRef [35] uses transfer rules to refine abstract datatypes and algorithms to executable code. Blanchette et al. [23] use parametricity to express well-formedness conditions for operators under which corecursion may appear in corecursive functions. In contrast, we derive a relational logic for reasoning about shallowly embedded programs from parametricity and apply representation independence to replace oracles in games by bisimilar ones.

6 Conclusion and Future Work

In this paper, we have integrated a language for monadic probabilistic functions with black-box access to oracles in higher order logic. Several examples demonstrate that cryptographic algorithms and security definitions can be expressed in this language. A rich equational theory and a relational logic support the formalisation of cryptographic arguments. The definitions and proofs have been mechanised using the proof assistant Isabelle/HOL.

Although our logic is similar to those in FCF and EasyCrypt, our approach is different: we derive the rules in a principled way from parametricity of the operators. This approach can help in finding the appropriate rules when new operations are introduced, and in proving them sound. Petcher and Morrisett, e.g., add a monadic map operation to FCF's language and an appropriate reasoning rule to their logic [43]. They show soundness for the rule by induction, which takes 22 proof steps. In our approach, their rule is an instance of parametricity of the map operator, which Isabelle's parametricity prover can establish automatically.

This work is motivated by the goal of formalising computational soundness results. Whether our framework scales to such large applications is still an open question. In our cryptographic examples, it compares favourably to the state of the art. Indeed, our initial attempts in formalising the CoSP framework [9] are encouraging that our approach will work. Therefore, the next steps will focus on formalising these results and improving proof automation.

Our framework cannot express efficiency notions such as polynomial runtime yet. Hence, asymptotic reasoning, which dominates in CS proofs, can be used only in limited ways. This is the flipside of the shallow embedding: As HOL functions are extensional, we cannot exploit HOL syntax in the reasoning. It seems possible to adapt Petcher's and Morrisett's axiomatic cost model [42], but the benefits are not yet clear. A less axiomatic alternative is to formalise a small programming language, connect it to HOL's functions and derive the bounds in terms of the operational semantics. Verypto [8] and the higher-order complexity analysis for Coq by Jouannaud and Xu [33] might be good starting points.

Beyond cryptographic arguments and computational soundness, our semantic domain of generative probabilistic values could be applied in different contexts. In previous work [38], we used a (less abstract) precursor to model interactive programs in HOL. The domain could also serve as a basis for formalising and verifying CryptoVerif or as a backend for the new EasyCrypt, as EasyCrypt's logic and module system resemble Isabelle's.

In this direction, it would be interesting to investigate whether gpvs admit a ccpo structure in which where the basic operations are monotone. Currently, corecursion and coinduction are available. The ccpo structure would allow us to additionally define functions recursively and use induction as proof principle. The problem is that limits must preserve discreteness of the subprobability distributions. It is not easy to move to arbitrary measure-theoretic distributions, as codatatypes in HOL require a cardinality bound on the functor through which they recurse; otherwise, they cannot be defined in HOL [28].

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